Boundary conditions in charge conjugate $sl(N)$ WZW theories

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We compute the representations ("NIM-reps") of the fusion algebra of $\widehat{sl}(N)$ which determine the boundary conditions of $\widehat{sl}(N)$ WZW theories twisted by the charge conjugation. This is done following two procedures, one of general validity, the other specific to the problem at hand. The problem is related to the classical problem of decomposition of the fundamental representations of $sl(N)$ onto representations of $B_l = so(2l+1)$ or $C_l = sp(2l)$ algebras. The relevant NIM-reps and their diagonalisation matrix are thus expressed in terms of modular data of the affine B or C algebras.

1. Introduction, notations and results

It is now well understood that the possible boundary conditions of a rational conformal field theory are determined by the set of non-negative integer valued matrix representations, or NIM-reps, of the fusion algebra of this theory [1]. In the present paper, we address the problem of determining NIM-reps and related data for those theories of WZW type, that are described by a modular invariant partition function twisted by complex conjugation

$$Z = \sum_\lambda \chi_\lambda(q)\chi_{\lambda^*}(\bar{q}) \,, \tag{1.1}$$

(notations will be made more precise below). To be specific, we restrict here to the $\widehat{sl}(N)$ current algebra. This exercise has the double merit of illustrating the power of certain methods of general application, and of exhibiting a nice algebraic pattern: indeed, it turns out that the problem is intimately connected with the classical problem of decomposing the representations of $sl(N)$ onto representations of the $B_l = so(2l+1)$ or $C_l = sp(2l)$ algebras, with $N = 2l$ or $2l+1$. This work generalises the previous results for $N = 3$ [2,1] and $N = 4$ [3].

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1.1. The $A_{N-1} = sl(N)$ and the affine $\widehat{sl}(N)$ algebras

To proceed, we need to introduce notations. As we are dealing with pairs of Lie algebras, we consistently use different types of labels for their representations etc. For the $\widehat{sl}(N)$ theories under study, weights will be denoted by Greek letters. At a given level k or shifted level $h = k + N$ these weights belong to the Weyl alcove

$$\mathcal{P}_{++}^{(A_{N-1},\,h)} := \{\lambda = \sum_{i=1}^{N-1} \lambda_i \Lambda_i \mid \lambda_i \geq 1\,, \sum_{i=1}^{N-1} \lambda_i \leq h - 1\}\ , \tag{1.2}$$

where Λ_i, $i = 1, \cdots, N-1$ are the $sl(N)$ fundamental weights. The Weyl vector is $\rho = \sum_{i=1} \Lambda_i$. The number of weights in $\mathcal{P}_{++}^{(A_{N-1},\,h)}$ equals $\binom{h-1}{N-1}$. The alcove is invariant under the action of C, the complex conjugation of representations, $C\ :\ \lambda = (\lambda_1, \cdots, \lambda_{N-1}) \mapsto \lambda^* = (\lambda_{N-1}, \cdots, \lambda_1)$, and of the $\mathbb{Z}_N$ automorphism σ, related to the cyclic symmetry of the affine Dynkin diagram of type A

$$\sigma(\lambda) = (h - \sum_{i=1}^{N-1} \lambda_i, \lambda_1, \cdots, \lambda_{N-2})\ . \tag{1.3}$$

Basic in our discussion is the symmetric, unitary matrix $S = (S_{\lambda\mu})$ of modular transformations. Under the action of C and σ,

$$S_{\lambda^*\mu} = S_{\lambda\mu^*} = (S_{\lambda\mu})^* \qquad S_{\sigma(\lambda)\mu} = e^{2\pi i \tau(\mu)/N}\, S_{\lambda\mu}\ , \tag{1.4}$$

where $\tau(\lambda) := \sum_{i=1}^{N-1} i(\lambda_i - 1)$ is the $\mathbb{Z}_N$ grading of weights –the "N-ality".

We want to find a set of matrices $\{n_\gamma\}_{\gamma \in \mathcal{P}_{++}^{(N;\,h)}}$ with non negative integer entries such that their matrix product reads

$$n_\lambda n_\mu = \sum_\nu N_{\lambda\mu}{}^\nu n_\nu \tag{1.5}$$

where $N_{\lambda\mu}{}^\nu$ are the fusion matrices of the $\widehat{sl}(N)$ theory at that level. The n_λ must satisfy Cardy consistency condition

$$n_{\lambda a}{}^b = \sum_{j \equiv j(\mu)\,,\ \mu \in \mathrm{Exp}\,^{(h)}} \frac{S_{\lambda\mu}}{S_{\rho\mu}}\, \psi_a^j\, \psi_b^{j\,*} \tag{1.6}$$

with ψ the unitary matrix diagonalising them; $j = j(\mu)$ labels a proper choice of basis. Their eigenvalues are thus of the form $\chi_\lambda(\mu) := S_{\lambda\,\mu}/S_{\rho\,\mu}$, S the modular matrix, and are specified by the weights μ labelling the *diagonal* terms in (1.1), called "exponents". In the case under study, the exponents are the real, i.e. self-conjugate, weights $\mu = \mu^*$ in the alcove. Depending on the parity of N, those have a different structure:

$$\mathrm{Exp}^{(h)} \ni \mu = \begin{cases} (m_1, \cdots, m_l, m_l, \cdots, m_1)\,, \quad 2\sum_{i=1}^{l} m_i \le h-1 & \text{if } N = 2l+1 \\ (m_1, \cdots, m_{l-1}, m_l, m_{l-1}, \cdots, m_1)\,, \quad 2\sum_{i=1}^{l-1} m_i + m_l \le h-1 & \text{if } N = 2l \end{cases} \tag{1.7}$$

and their number is

$$|\mathrm{Exp}^{(h)}| = \ \#\text{real weights} = \begin{cases} \binom{\lfloor \frac{h-1}{2} \rfloor}{l} & \text{if } N = 2l+1 \\ \binom{\lfloor \frac{h}{2} \rfloor}{l} + \binom{\lfloor \frac{h-1}{2} \rfloor}{l} & \text{if } N = 2l \end{cases} \tag{1.8}$$

In general, the NIM-rep matrices satisfy $n_\lambda^T = n_{\lambda^*}$; in the present case, because of the reality of the exponents μ, their eigenvalues $\chi_\lambda(\mu)$ are real and satisfy $\chi_\lambda(\mu) = \chi_{\lambda^*}(\mu)$ and one concludes that the matrices n_λ are symmetric. Moreover, because a real weight μ has a N-ality τ equal to $0 \mod N$, resp. 0 or $N/2 \mod N$, for N odd, resp. even, eq. (1.4) implies that n_λ is only a function of the orbit of λ under σ, resp. σ^2. As usual, it is sufficient to find the generators $n_{\bar\Lambda_i} = n_{\bar\Lambda_{N-i}}$ associated with the fundamental weights to fully determine the NIM-rep. If the matrices $n_\lambda = \left(n_{\lambda\, a}{}^b\right)$ are regarded as adjacency matrices of graphs, it is natural to refer to the labels a, b of their entries as *vertices*. On the latter, we do not know much a priori, besides that their number equals the number of exponents (1.8). The set of vertices is denoted by $\mathcal{V}$.

Along with the NIM-rep matrices n_λ, we are also interested in finding a related set of matrices $\hat N_a = \left(N_{ba}{}^c\right)$, satisfying $\hat N_a \hat N_b = \sum_c \hat N_{ab}{}^c \hat N_c$ and

$$n_\lambda \hat N_a = \sum_{b\in\mathcal{V}} n_{\lambda a}{}^b \hat N_b \ . \tag{1.9}$$

These matrices, associated with the vertices of the graph, span the "graph algebra", which in this particular case is commutative. In particular, the set includes the unit matrix attached to a special vertex denoted 1 : $\hat N_1 = I$. Then the previous relation evaluated for $a = 1$ gives

$$n_\lambda = \sum_{b\in\mathcal{V}} n_{\lambda 1}{}^b \hat N_b \ , \tag{1.10}$$

i.e. the NIM-rep matrices are ≥ 0 integer linear combinations of the $\hat N$. The matrix ψ in (1.6) diagonalises both n and $\hat N$ and (1.10) can be also rewritten as

$$\chi_\lambda(\mu) = \sum_{a\in\mathcal{V}} n_{\lambda 1}{}^a \hat\chi_a(j(\mu)) \ , \tag{1.11}$$

where $\hat{\chi}_a(j) = \psi_a^j/\psi_1^j$, $j \in \text{Exp}^{(h)}$, are the eigenvalues of $\hat{N}_a$.

In the present context the equations (1.9), (1.10) have a natural group theoretic interpretation. This is clear already in the simplest case $N = 3$ [1]. The reality of the exponents (1.7) implies that they can be identified with an integrable weight $\mu \to j(\mu)$ of $\widehat{sl}(2)$ at a related level. Then depending on the parity of h, the coefficients $n_{\lambda 1}{}^a$ originate from different patterns of decomposition of the representations of $sl(3)$ into those of $sl(2)$. Namely the graphs are determined by the fundamental NIM-rep which is either $n_{\Lambda_1+\rho\, 1}{}^a = 1+\delta_{a\, 2w}$, or $n_{\Lambda_1+\rho\, 1}{}^a = \delta_{a\, 3w}$, with w the $sl(2)$ fundamental weight, thus reflecting the two ways of decomposing the 3- dimensional $sl(3)$ representation. As we shall see, this example is the first in the series for odd N, with C_l and B_l taking over the rôle of $sl(2)$ for h even or odd respectively. Similarly the formula (1.9) is interpreted for large h as the decomposition rule of the tensor product of an irreducible representation (irrep) of A_{N-1} times an irrep of the subalgebra into irreps of the subalgebra. The "branching coefficients" interpretation of the NIM-reps and the equations (1.5),(1.10) has been discussed also in the context of the discrete subgroups of $SU(2)$. See also [4] for a related recent discussion. Given the diagonalisation matrix ψ_a^j one can compute as well the structure constants of the algebra dual to the graph algebra, the Pasquier algebra, which admits important physical interpretations [1].

1.2. B and C algebras

We now briefly introduce relevant notations for the Lie algebras B_l and C_l and their affine extensions $B_l^{(1)}$ and $C_l^{(1)}$.

In the B_l algebra, we denote the integrable weights by Latin letters, keeping however the Greek w_i for the fundamental weights and $\bar{\rho}$ - for the Weyl vector. As the dual Coxeter number is $h^\vee = 2l-1$, the Weyl alcove at level k is the set

$$\mathcal{P}_{++}^{(B_l,h)} = \{m = \sum_{i=1}^{l} m_i w_i \mid m_i \geq 1,\ m_1 + 2\sum_{i=2}^{l-1} m_i + m_l \leq h-1\}\ , \tag{1.12}$$

where the notation h is again used for the shifted level $h = k + 2l - 1$. The number of integrable weights is $|\mathcal{P}_{++}^{(B_l,\,h)}| = \binom{\lfloor \frac{h+2}{2} \rfloor}{l} + 2\binom{\lfloor \frac{h+1}{2} \rfloor}{l} + \binom{\lfloor \frac{h}{2} \rfloor}{l}$. These weights are graded according to a $\mathbb{Z}_2$ grading $\tau(m) := m_l - 1 \bmod 2$ and the $\tau = 0$ weights label a subalgebra of the Verlinde fusion algebra. The $\mathbb{Z}_2$ automorphism of the affine B_l Dynkin diagram acts on the weights in the alcove as $\sigma(m) = (h - m_1 - 2\sum_{i=2}^{l-1} m_i - m_l\ , m_2, \ldots, m_l)$.

For the C_l algebra, we use parallel notations: fundamental weights are again denoted w_i, $i=1,\cdots,l$; the dual Coxeter number is $h^\vee = l+1$ whence the shifted level $h=k+l+1$; the Weyl alcove reads

$$\mathcal{P}_{++}^{(C_l,h)} = \{m=\sum_{i=1}^{l} m_i w_i \mid m_i \geq 1,\ \sum_{i=1}^{l} m_i \leq h-1\} \ ; \tag{1.13}$$

the number of weights in the alcove is $|\mathcal{P}_{++}^{(C_l,h)}| = \binom{h-1}{l}$; the $\mathbb{Z}_2$ grading reads $\tau(m) := \sum_{i=1}^{l} i(m_i-1) \bmod 2$. The $\mathbb{Z}_2$ automorphism of the affine C_l Dynkin diagram acts on the weights in the alcove as $\sigma(m) = (m_{l-1},\ldots,m_1\ ,\ h-\sum_{i=1}^{l} m_i)$.

The S matrices of B and C type are real and satisfy a $\mathbb{Z}_2$ analog of the σ symmetry property (1.4).

1.3. Results

We may summarise our results as follows. In general the eigenvalues in the r.h.s. of (1.11) are expressed by the modular matrices S of B_l or C_l

$$\hat{\chi}_a(j) = \frac{S_{aj}}{S_{1j}}\ , \tag{1.14}$$

in which the weights of B or C algebras label both the graph vertices $a \in \mathcal{V}$ and the basis $j=j(\mu)$ into (1.6), related to a projection of the set of exponents (1.7) to the B or C alcoves; the vertex $a=1$ in (1.14) is identified with the B or C Weyl vector $\bar{\rho}$, i.e. the shifted weight of the identity representation.
The situation depends on the parities of N and of the shifted level h.

1) For $N=2l+1$, h even:

The set of exponents $\mathrm{Exp}^{(h)}$ (1.7) is identified with the C_l integrable alcove $\mathcal{P}_{+,+}^{(C_l,\frac{h}{2})}$

$$\mathcal{P}_{+,+}^{(C_l,\frac{h}{2})} \ni j(\mu) = (m_1,m_2,\ldots,m_l) \Leftrightarrow \mu = (m_1,\ldots,m_l,m_l,\ldots,m_1) \in \mathrm{Exp}^{(h)} \tag{1.15}$$

and the same alcove parametrises as well the set of graph vertices $\mathcal{V} \equiv \mathcal{P}_{+,+}^{(C_l,\frac{h}{2})}$. The Pasquier and graph algebras are identical and coincide with the C_l Verlinde fusion algebra, $\hat{N}_a = N_a$. Accordingly ψ_a^j in (1.6) is provided by the C_l modular matrix S,

$$\psi_a^j = S_{aj}\ , \qquad a,j \in \mathcal{P}_{+,+}^{(C_l,\frac{h}{2})}\ . \tag{1.16}$$

The decomposition formula (1.10) for the fundamental NIM-reps reads

$$n_{\Lambda_i+\rho} = \sum_{k=0}^{i} \hat{N}_{\omega_{i-k}+\bar{\rho}}\,, \qquad i = 1, 2, \ldots, l\,. \tag{1.17}$$

For $h = 2l + 2$ (1.17) degenerates to $n_{\Lambda_i+\rho} = 1$ for all i.

2) For $N = 2l + 1$, h odd:

The set of exponents (1.7) is identified with a subset of the alcove $\mathcal{P}_{+,+}^{(B_l,h)}$

$$\mathrm{Exp}^{(B)} = \{\mathcal{P}_{+,+}^{(B_l,h)} \ni j = (m_1, m_2, \ldots, m_l)\,|\,\tau(j) = 1\,,\, m_1 < \frac{h - m_l}{2} - \sum_{i=2}^{l-1} m_i\,\}\,, \tag{1.18}$$

where (note $\tau(j) + 1 = m_l = 0 \bmod 2$)

$$\mathrm{Exp}^{(B)} \ni j = (m_1, m_2, \ldots, m_l) \Leftrightarrow \mu = (m_1, \ldots, \frac{m_l}{2}, \frac{m_l}{2}, \ldots, m_1) \in \mathrm{Exp}^{(h)}\,.$$

Another subset of $\mathcal{P}_{+,+}^{(B_l,h)}$ parametrises the vertices

$$\mathcal{V} = \{\mathcal{P}_{+,+}^{(B_l,h)} \ni a = (m_1, m_2, \ldots, m_l)\,|\,\tau(a) = 0\,,\; m_1 < \frac{h - m_l}{2} - \sum_{i=2}^{l-1} m_i\,\}\,. \tag{1.19}$$

The eigenvector matrix in (1.6) is given by

$$\psi_a^j = 2\, S_{aj}\,, \quad a \in \mathcal{V}\,, j \in \mathrm{Exp}^{(B)}\,. \tag{1.20}$$

As empirically observed, there exists a basis (i.e., a preferred correspondence of the two sets of indices $\mathcal{V}$, $\mathrm{Exp}^{(B)}$, in which the matrix ψ_a^j is symmetric and hence the Pasquier algebra is identical to the graph algebra. The matrices $\hat{N}_a$ are expressed via the B_l Verlinde fusion matrices,

$$\hat{N}_{ab}{}^{c} = N_{ab}{}^{c} - N_{ab}{}^{\sigma(c)} \geq 0\,, \qquad a, b, c \in \mathcal{V}\,. \tag{1.21}$$

The fundamental NIM-reps are

$$n_{\Lambda_i+\rho} = \hat{N}_{\omega_i+\bar{\rho}}, \quad i = 1, 2, ..., l-1\,, \qquad n_{\Lambda_l+\rho} = \hat{N}_{2\omega_l+\bar{\rho}}\,. \tag{1.22}$$

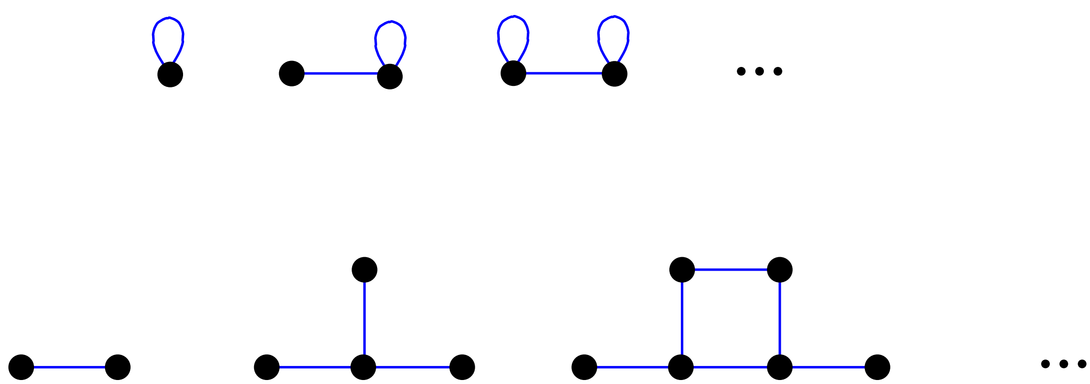
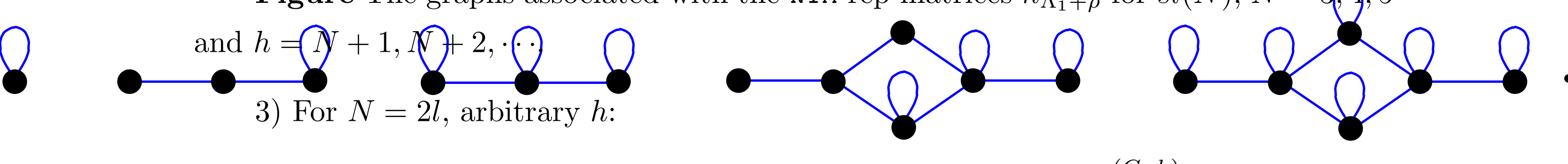


Figure The graphs associated with the $\texttt{NIM}$-rep matrices $n_{\Lambda_1+\rho}$ for $sl(N)$, $N = 3, 4, 5$ and $h = N+1, N+2, \cdots$.



3) For $N = 2l$, arbitrary h:

The set of exponents (1.7) is identified with a subset of the alcove $\mathcal{P}_{+,+}^{(C_l,h)}$

$$\mathrm{Exp}^{(C)} = \{\mathcal{P}_{+,+}^{(C_l,h)} \ni j = (m_1, m_2, \dots, m_l) |\ m_1, \dots, m_{l-1} - \text{even}\}\,, \qquad (1.23)$$

$$\mathrm{Exp}^{(C)} \ni j = (m_1, m_2, \dots, m_l) \Leftrightarrow \mu = (\frac{m_1}{2}, \dots, \frac{m_{l-1}}{2}\,, m_l\,, \frac{m_{l-1}}{2}, \dots, \frac{m_1}{2}) \in \mathrm{Exp}^{(h)}\,.$$

A subset of $\mathcal{P}_{+,+}^{(C_l,h)}$ parametrises the vertices

$$\mathcal{V} = \mathcal{P}_{+,+}^{(C_l,\lfloor\frac{h}{2}\rfloor+1)} \cup \sigma_1(\mathcal{P}_{+,+}^{(C_l,\lfloor\frac{h}{2}\rfloor+1)}) \subset \mathcal{P}_{+,+}^{(C_l,h)} \tag{1.24}$$

where

$$\sigma_1(m_1,\ldots,m_l) := (h - m_1 - 2\sum_{i=2}^{l} m_i, m_2, m_3, \ldots, m_l)\,. \tag{1.25}$$

For h odd (1.24) is a disjoint union of two subsets of $\mathcal{P}_{+,+}^{(C_l,h)}$ of the same cardinality. The eigenvector matrix ψ_a^j is expressed by the C_l modular matrix S

$$\psi_a^j = (\sqrt{2})^{l-1}\, S_{aj}\,, \qquad a \in \mathcal{V}\,,\ j \in \mathrm{Exp}^{\,(C)}\,, \tag{1.26}$$

Empirical data suggest that in general ($l > 2$), ψ in (1.26) is not symmetrisable for h even.

For h odd the graph algebra matrices $\hat{N}_a$ are nonnegative, while for h even they may have signs. (For $h = 2l+1$ the graph algebra is isomorphic to $\mathbb{Z}_2$.) The fundamental NIM-reps are

$$n_{\Lambda_i+\rho} = \sum_{m=0}^{\lfloor i/2\rfloor} \hat{N}_{\omega_{i-2m}+\bar{\rho}}\,, \qquad i = 1,2,\ldots,l\,, \tag{1.27}$$

where $\omega_0 := 0$, i.e., $\hat{N}_{\bar{\rho}} = 1$. The simplest example in this series is the case $\widehat{sl}(4)$ and (1.27) reproduces the graphs displayed in [3], see the Figure. In this particular example the graph matrices are expressed by the C_l Verlinde matrices as in (1.21), non-negative for h odd.

The group theoretic approach selects one of the possible solutions for $n_{\lambda 1}{}^1$, $\lambda \in \mathcal{P}_{+,+}^{(A_{N-1},h)}$, namely the one in which $n_{\lambda 1}{}^1$ is the classical multiplicity of the identity representation of C_l or B_l in the irreducible representation of A_{N-1} of highest weight λ.

The rest of this note will review the two routes which led us to these results.

2. Decompositions of irreps – classical consideration and quantisation

The reality condition on the exponents in (1.7) leads to a $\mathbb{Z}_2$ folding of the integrable alcove, or effectively, of the set of A_{N-1} roots, thus suggesting the relevance of the algebras of B- or C- type. Recall that the Verlinde matrix eigenvalues, or, fusion algebra characters, can be interpreted as classical characters, evaluated on a discrete subset of the Cartan group. The reduction of groups and their irreducible representations ("irreps") can be described in terms of the weight diagrams $\mathcal{G}_\lambda$ associated with the highest weights λ (in this "classical" discussion, unshifted), or, in other words, as a decomposition

$$\chi_{\lambda+\rho} = \sum n_{\lambda+\rho\,1}{}^a \hat{\chi}_a \,. \tag{2.1}$$

This involves their formal characters, i.e., sums of formal exponentials $\chi_{\lambda+\rho} = \sum_{\mu\in\mathcal{G}_\lambda} m_\mu\, e^\mu$ where $m_\mu \in \mathbb{Z}_{\geq 0}$ is the multiplicity of the weight μ in the weight diagram $\mathcal{G}_\lambda$ of the given finite dimensional irrep of highest weight λ. Thus the first step in solving our problem is the determination of the classical decomposition formulae, i.e., the coefficients in (2.1), (whence the identical notation as in (1.11)), by choosing a projection P of the weights of A_{N-1} and decomposing the weight diagram. In the A_2 example choosing $\mu \to P_1(\mu) = \mu_1 + \mu_2$ or $\mu \to P_2(\mu) = 2(\mu_1+\mu_2)$ leads to $P_1(\mathcal{G}_{(1,0)}) = \mathcal{G}_0 + \mathcal{G}_w$, or $P_2(\mathcal{G}_{(1,0)}) = \mathcal{G}_{2w}$. It is sufficient to establish these relations for the fundamental representations of $sl(n)$; to recover the remaining ones, we use the standard Pieri formulae. The second step is based on the alternative interpretation of the classical characters as group characters, $\chi_i(H)$, evaluated on the Cartan subgroup H, common to $SL(N)$ and its subgroup $SO(2l{+}1)$ or $SP(2l)$. Reducing to a subgroup means that typically the symmetry of the formal characters is enlarged, e.g., we can identify the weight diagram of a representation and its complex conjugate, and thus in particular restrict to half the fundamental A_{N-1} characters. The quantisation procedure leading to the fusion characters amounts to evaluating the formal exponentials on rational points $e^\lambda(\frac{2\pi i\mu}{h}) = e^{\frac{2\pi i\,\langle\lambda,\mu\rangle}{h}}$, $\mu \in \mathcal{P}_{+,+}^{(h)}$. This selects an embedding of the range $\mathrm{Exp}^{(h)}$ of exponents in an integrable alcove of the smaller algebra. E.g., for A_2, $\lambda \in \mathcal{G}_{(1,0)}$ and $\mu = (m,m) \in \mathcal{P}_{+,+}^{(A_2,h)}$ one has $\langle\lambda,\mu\rangle^{(A_2)} = 2\langle P_1(\lambda), m\omega\rangle^{(A_1)} = \langle P_2(\lambda), m\omega\rangle^{(A_1)}$, which allows to identify $\mathrm{Exp}^{\,(h)}$ with the A_1 alcove $\mathcal{P}_{+,+}^{(A_1,\frac{h}{2})}$ for h even, or with a factor $\mathcal{P}_{+,+}^{(A_1,h)}/\mathbb{Z}_2$ for odd h.

Performing these two steps leads to the eigenvalue decomposition formula (1.11) which we shall use for the fundamental weights $\lambda = \Lambda_i$. The "quantisation" of the classical characters is equivalent to further enlarging their symmetry, from the corresponding Weyl $\overline{W}$ to affine Weyl groups. This affects the general classical decomposition coefficients in (2.1), which are expressed according to (1.6) in terms of these characters $n_{\lambda 1}{}^a = \sum_{\mu\in\mathrm{Exp}^{(h)}} (\psi_1^j)^2\, \chi_\lambda(\mu)\, \hat{\chi}_a(j(\mu))$. (The factor $(\psi_1^j)^2$ here is the rational analog of the factor in the Haar measure in the corresponding integral classical formula.) Since in what follows we shall restrict to fundamental weights, the coefficients in (1.11) precisely coincide with their classical analogs in (2.1) in all but the smallest value of h in the cases 1) and 3) in section 1.3. E.g., in our example, $\chi_{\Lambda_1+\rho}((m,m)) = 1 + \hat\chi_{2w}(m)$, $m\omega \in \mathcal{P}_{+,+}^{(A_1,\frac{h}{2})}$, or $\chi_{\Lambda_1+\rho}((m,m)) = \hat\chi_{3w}(m)$, $m\omega \in \mathcal{P}_{+,+}^{(A_1,h)}$, $m \le \frac{h-1}{2}$, respectively. In the case $h=4$ even the weight $2w$ is on the shifted by $\bar\rho$ "wall" $m = \frac{h}{2}$ of the alcove $\mathcal{P}_{+,+}^{(A_1,\frac{h}{2})}$ and hence the character $\hat\chi_{2w}$ and the multiplicity ${n_{\Lambda_1+\rho\,1}}^{2w}$ vanish.

Finally we recover the matrix equivalent (1.10) of this formula, by finding an appropriate range $\mathcal{V}$ of the same cardinality as $\mathrm{Exp}^{(h)}$, and an unitary matrix ψ.

2.1. Details and elements of proofs

Let μ be a weight in the weight lattice of $sl(N)$. Define the projections to weights of B_l or C_l

$$\begin{aligned}
\mu \to P_1(\mu) &= (\mu_1+\mu_{2l},\mu_2+\mu_{2l-1},\dots,\mu_l+\mu_{l+1})^{(C_l)} = \sum_{m=1}^{l} \langle P_1(\mu)\,,\,\beta_m^\vee\rangle^{(C_l)}\, w_m \\
&= \sum_{m=1}^{l} \langle \mu\,,\,\alpha_m+\alpha_{2l+1-m}\rangle^{(A_{2l})}\, w_m \\
\Rightarrow P_1(\Lambda_i) &= w_i\,,\ i=1,\dots,l\,,\quad P_1(\alpha_i) = \frac{\beta_i^\vee}{2}\,,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\mu \to P_2(\mu) &= (\mu_1+\mu_{2l},\mu_2+\mu_{2l-1},\dots,2(\mu_l+\mu_{l+1}))^{(B_l)} = \sum_{m=1}^{l} \langle P_2(\mu)\,,\,\beta_m^\vee\rangle^{(B_l)}\, w_m \\
&= \sum_{m=1}^{l-1} \langle \mu\,,\,\alpha_m+\alpha_{2l+1-m}\rangle^{(A_{2l})}\, w_m + 2\langle \mu\,,\,\alpha_l+\alpha_{l+1}\rangle^{(A_{2l})}\, w_l \\
\Rightarrow P_2(\Lambda_i) &= w_i\,,\ i=1,\dots,l-1\,,\ P_2(\Lambda_l) = 2w_l\,,\quad P_2(\alpha_i) = \beta_i\,,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
\mu \to P_3(\mu) &= (\mu_1+\mu_{2l-1},\mu_2+\mu_{2l-2},\dots,\mu_l)^{(C_l)} = \sum_{m=1}^{l} \langle P_3(\mu)\,,\,\beta_m^\vee\rangle^{(C_l)}\, w_m \\
&= \sum_{m=1}^{l-1} \langle \mu\,,\,\alpha_m+\alpha_{2l+1-m}\rangle^{(A_{2l-1})}\, w_m + \langle \mu\,,\,\alpha_l\rangle^{(A_{2l-1})}\, w_l \\
\Rightarrow P_3(\Lambda_i) &= w_i\,,\ i=1,\dots,l\,,\quad P_3(\alpha_i) = \beta_i\,.
\end{aligned} \tag{2.4}$$

Here ($\beta_i^\vee$) β_i stand for the (dual) roots of B_l or C_l. Clearly $P_m(\Lambda_i) = P_m(\Lambda_{N-i})$. With the exception of $P_2(\Lambda_l) = 2w_l$ (so that $\tau(P_2(\Lambda_i)) = 0\ \forall i = 1,\dots,l$) the maps P_i relate the highest weights of the fundamental representations of the pairs of algebras. (There is one more natural projection $A_{2l-1} \to B_l$, s.t. $\alpha_i \to \beta_i^\vee$, which however is not useful for our purposes as it produces solutions of (1.5) with some negative entries.) Applying furthermore these maps on the weights λ in the weight diagrams $\mathcal{G}_{\Lambda_i}$ of the fundamental irreps of A_{N-1} one obtains the classical result

Proposition: (Classical decomposition formulae)

$$P_1(\mathcal{G}^{(A_{2l-1})}_{\Lambda_i}) = \oplus_{p=0}^{i}\, \mathcal{G}^{(C_l)}_{\omega_{i-p}} \tag{2.5}$$

$$P_2(\mathcal{G}^{(A_{2l})}_{\Lambda_i}) = \mathcal{G}^{(B_l)}_{\omega_i}\,,\ i=1,\dots,l-1\,; \qquad P_2(\mathcal{G}^{(A_{2l})}_{\Lambda_l}) = \mathcal{G}^{(B_l)}_{2\omega_l} \tag{2.6}$$

$$P_3(\mathcal{G}^{(A_{2l-1})}_{\Lambda_i}) = \oplus_{p=0}^{\lfloor i/2\rfloor}\, \mathcal{G}^{(C_l)}_{\omega_{i-2p}} \tag{2.7}$$

The next step is to interpret these decomposition formulae as relations for the evaluated characters. The characters in the l.h.s. are evaluated on the exponent subset $\mathrm{Exp}^{(h)}$ of the A_{2l} (or A_{2l-1}) integrable alcove. One finds that for $\lambda \in \mathcal{G}^{(A_{2l})}_{\Lambda_i}$ and $\mu \in \mathrm{Exp}^{(h)} \subset \mathcal{P}^{(A,h)}_{+,+}$

$$\begin{aligned} \langle\lambda,\mu\rangle^{A_{2l}} &= 2\langle P_1(\lambda),(\mu_1,\dots,\mu_l)\rangle^{C_l} = 2\langle P_1(\lambda), j(\mu)\rangle^{C_l}\,, \qquad j(\mu) = \frac{P_1(\mu)}{2}\,,\\ \langle\lambda,\mu\rangle^{A_{2l}} &= \langle P_2(\lambda),(\mu_1,\dots,\mu_{l-1},2\mu_l)\rangle^{B_l} = \langle P_2(\lambda), j(\mu)\rangle^{B_l}\,, \qquad j(\mu) = \frac{P_2(\mu)}{2}\,. \end{aligned} \tag{2.8}$$

In the first case the integrability condition for $\mu \in \mathrm{Exp}^{(h)}$ is equivalent to the C_l integrability condition for $j(\mu)$ at shifted level $\lfloor \frac{h}{2} \rfloor$. For h even, taking into account the factor 2, one arrives at (1.15). The classical decomposition formula (2.5) determines the coefficients in (1.11) for the fundamental characters χ_{Λ_i}. These characters generate the C_l Verlinde fusion algebra which is diagonalised by the modular matrix S and thus we recover the case 1) of sect. 1.3.

For h odd we use instead the second equality in (2.8). The map P_2 embeds the exponent set $\mathrm{Exp}^{(h)}$ into a subset of the B_l integrable alcove $\mathcal{P}^{(B_l,h)}_{+,+}$, the one described in (1.18), i.e., a $\mathbb{Z}_2$ - factor of the $\tau = 1$ subset. The eigenvalues $\{\chi_{w_1+\rho}(j)\,,\, \chi_{w_2+\rho}(j)\,, \dots \chi_{w_{l-1}+\rho}(j)\,,\, \chi_{2w_l+\rho}(j)\}$, all labelled by weights of $\tau = 0$, generate an algebra $\hat{N}_a$ with a in the range (1.19). Using the standard properties of the modular matrix S it is easily derived that this subalgebra is diagonalised by the matrix in (1.20), indeed

$$\delta_{jj'} = \sum_m S_{mj} S_{ij'} = 2 \sum_{\tau(m)=0} S_{mj} S_{mj'} = 4 \sum_{a\in\mathcal{V}} S_{aj}\, S_{aj'}\,, \qquad j,j' \in \mathrm{Exp}^{(\mathrm{B})}\,.$$

In the first step we have used that there are no σ fixed points in $\mathrm{Exp}^{(\mathrm{B})}$, while in the second, that $S_{\sigma(m)j} = -S_{mj}$ for $j \in \mathrm{Exp}^{(\mathrm{B})}$, in particular $S_{m_0 j} = 0$ for the fixed points

$m_0 = \sigma(m_0)$. To prove the expression (1.21) for the structure constants we proceed similarly as above but this time obtain the exponent region $\mathrm{Exp}^{(\mathrm{B})}$ by factorising the B_l alcove over the $\mathbb{Z}_2$ symmetry of the σ automorphism. Namely using that $S_{a\sigma(j)} = S_{aj}$, for $a \in \mathcal{V}$ (since $\tau(a) = 0$), we split the summation in the Verlinde formula for N_{ab}^c with all $a, b, c \in \mathcal{V}$ according to

$$N_{ab}^c = \Big(2 \sum_{m\,,\,\langle m\,,\,\alpha_1\rangle < \langle \sigma(m)\,,\,\alpha_1\rangle} + \sum_{m\,,\,\langle m\,,\,\alpha_1\rangle = \langle \sigma(m)\,,\,\alpha_1\rangle} \Big) \frac{S_{am} S_{bm} S_{cm}}{S_{1m}}\,.$$

Note that since h is odd only points m with $\tau(m) = 0$ may contribute to the second term. We apply this formula to the difference $N_{ab}^c - N_{ab}^{\sigma(c)}$, using that $S_{cm} - S_{\sigma(c)m} = S_{cm}(1 - (-1)^{\tau(m)}) = 0$ for $\tau(m) = 0$, while it gives a factor 2 for $m \in \mathrm{Exp}^{(\mathrm{B})}$, and hence we get (1.21). This reproduces the case 2) of sect. 1.3.

Finally one finds for $\lambda \in \mathcal{G}_{\Lambda_i}^{(A_{2l-1})}$ and $\mu \in \mathrm{Exp}^{(h)} \subset \mathcal{P}_{+,+}^{(A,h)}$

$$\langle \lambda, \mu \rangle^{A_{2l-1}} = \langle P_3(\lambda), (2\mu_1, 2\mu_2, \dots, \mu_l) \rangle^{C_l} = \langle P_3(\lambda), j(\mu) \rangle^{C_l}\,, \qquad j(\mu) = P_3(\mu)\,. \tag{2.9}$$

This determines the range (1.23) for which the equality of eigenvalues implied by the classical formula (2.7) holds. However the cardinality of this range, cf. (1.8), is considerably smaller than the cardinality of the full alcove $\mathcal{P}_{+,+}^{(C_l,h)}$ and this makes the last case 3) more non-trivial. The set $\mathcal{V}$ in (1.24) is obtained by some $(\mathbb{Z}_2)^{l-1}$ folding of the C_l alcove $\mathcal{P}_{+,+}^{(C_l,h)}$. Define recursively a sequence of involutive maps and a sequence of subsets of the C_l alcove

$$\begin{gathered}\sigma_s(m) = (m_{s-1}, \dots, m_1\,,\, h - \sum_{k=1}^{s} m_k - 2 \sum_{k=s+1}^{l} m_k\,,\, m_{s+1}, \dots m_l)\,,\ m \in \mathcal{A}_{s+1} \\ \mathcal{A}_s = \{m \in \mathcal{A}_{s+1} |\ m_s = \langle m, \alpha_s^\vee \rangle \ < \ \langle \sigma_s(m), \alpha_s^\vee \rangle \ \}\,, \quad s = l, \dots, 1\,,\ \mathcal{A}_{l+1} = \mathcal{P}_{+,+}^{(C_l,h)}\end{gathered} \tag{2.10}$$

so that $\sigma_l = \sigma$. The set (1.24) is easily seen to be $\mathcal{V} = \mathcal{A}_2$ by comparing the inequalities that define both.

The maps σ_s come from the horizontal projection of the action of the elements $w^{(s)} = t_{w_s}\, \bar{w}^{(s)}$ in the extended affine Weyl group $\tilde{W}$; here t_{w_s} is an affine translation, while $\bar{w}^{(s)}$ is the unique element in the Weyl group $\overline{W}$, which keeps invariant the subset of roots $\{\alpha_1, \alpha_2, \dots, \alpha_l\,, -\alpha^{(s)}\}$, where $\alpha^{(s)} = e_1 + e_{s+1}$; e_i are the orthogonal vectors $\langle e_i, e_j \rangle = \frac{1}{2}\, \delta_{ij}$, $w_j = \sum_{i=1}^{j} e_i$, and by convention $e_{l+1} := e_1$. The transformations σ_s lead to symmetries of the C_l modular matrix, extending the analogous relation for $\sigma = \sigma_l$:

$$S_{\sigma_s(a)\, j} = (-1)^{s(j_l - 1) + \lfloor s/2 \rfloor}\, S_{a\, j}\,, \quad a \in \mathcal{A}_{s+1}\,,\ j \in \mathrm{Exp}^{\,(C)}\,. \tag{2.11}$$

Indeed the sign $(-1)^{s+\lfloor s/2\rfloor}$ comes, after reordering in the Weyl formula for S, from the parity of the element $\bar w^{(s)}: (e_1,e_2,\dots,e_l)\to(-e_s,-e_{s-1},\dots,-e_1,e_{s+1},\dots,e_l)$, while for $j\in \mathrm{Exp}^{(C)}$ we have $e^{-2\pi i\langle w_s, w(j)\rangle} = e^{-2\pi i j_l \langle w_s, w(\sum_{i=1}^l e_i)\rangle} = (-1)^{s j_l}$ for any $w\in\overline{W}$. The relation (2.11) implies that $\chi_{\sigma_s(1)}(j)$ are the eigenvalues of a simple current, i.e. $\chi_{\sigma_s(1)}(j)\chi_a(j)=\chi_{\sigma_s(a)}(j)$.

Furthermore a relation stronger than (2.11) holds for the particular points $m\in\mathcal{A}_{s+1}$ satisfying $m_s=(\sigma_s(m))_s = h-\sum_{p=1}^s m_p - 2\sum_{p=s+1}^l m_p$ for $1\le s\le l$, namely

$$S_{mj}=0\,, \qquad \text{if} \quad m\in\mathcal{A}_{s+1}\,, \sum_{p=1}^l m_p = h-\sum_{p=s}^l m_p\,,\ j\in \mathrm{Exp}^{(C)}\,. \tag{2.12}$$

Proof: If m has the property in (2.12) then $w^{(1,s)}(m)=m$ where $w^{(1,s)}\in\overline{W}$ is the Weyl reflection which maps $(e_1,e_s)\to(-e_s,-e_1)$ (for $s=1$, $e_1\to -e_1$), keeping all the remaining e_i unchanged. The Weyl group $\overline{W}$ splits into pairs $\{w, w^{(1,s)}w\}\in\overline{W}/\mathbb{Z}_2$ and the contribution to S_{mj} of each pair is zero.

The relations (2.11), (2.12) imply that the matrix defined in (1.26) is unitary. Indeed for $j+j'=0$ mod 2 we have $(1+(-1)^{s(j_l+j'_l)})=2$ and using (2.11) and (2.12) one proves recursively that

$$\sum_{a\in\mathcal{V}}\psi_a^j\psi_a^{j'} = 2^{l-2}\sum_{m\in\mathcal{A}_3} S_{mj}S_{mj'} = \ldots = \sum_{m\in\mathcal{A}_{l+1}} S_{mj}S_{mj'} = \delta_{jj'}\,.$$

If instead $j_l+j'_l=1$ mod 2, hence $j\neq j'$, one may use the decomposition $\mathcal{A}_2=\mathcal{A}_1\cup\sigma_1(\mathcal{A}_1)$ to write $\sum_{a\in\mathcal{V}}\psi_a^j\psi_a^{j'} = \sum_{a\in\mathcal{A}_1}\psi_a^j\psi_a^{j'}(1+(-1)^{j_l+j'_l})=0$ using (2.11) and (for h even) (2.12) for $s=1$. This completes the case 3). We note that empirical data suggest that there are formulae generalising (1.21), which represent the $\hat N_a$ by an alternating sum of C_l Verlinde fusion multiplicities.

3. Xu algorithm

In this section we sketch a method for the construction of the NIM-rep starting from a limited information. Although this method has been proposed and used in a more abstract form already a while ago [5,6], its algorithmic and systematic character may not have been stressed enough. In this approach, the fundamental data is the set of representations λ which give a non vanishing ${n_{\lambda 1}}^1$. The fact that these data play a central rôle has been discussed before [1,6] and stressed also recently in the category theory approach to these questions [7,8]. The meaning of these data is best understood in block diagonal theories, since there, the vertex denoted 1 corresponds to the block of the identity representation, and ${n_{\lambda 1}}^1$ thus tells us which representations appear in that block.

One may now proceed in three steps

1. The data ${n_{\nu 1}}^1$ are subject to a constraint coming from (1.6) and the unitarity of the S matrix

$$\sum_{\nu\in\mathcal{P}_{++}} \Big(\sum_{\mu\in \mathrm{Exp}^{(h)}} S_{\lambda\mu} S^*_{\nu\mu} - \delta_{\lambda\nu} \Big) {n_{\nu 1}}^1 = 0 \ . \tag{3.1}$$

It is thus helpful to have a good Ansatz for the particular matrix element ${n_{\nu 1}}^1$ and to check that it passes the test of (3.1). As explained in sect 1.1, in the present problem, ${n_{\nu 1}}^1$ is only a function of the C- and $\sigma^{\#}$-orbit of ν, with $\# = 1, 2$ for N odd, even, and this reduces greatly the number of unknowns.

2. One then regards the yet undetermined matrices n_λ as forming a set of vectors and one makes use of the `NIM`-rep property (1.5), taking its $1,1$ matrix element, to define a symmetric scalar product between these vectors according to

$$\langle n_\lambda, n_\mu \rangle := (n_\lambda . n_{\mu^*})_1{}^1 = \sum_a {n_{\lambda 1}}^a {n_{\mu 1}}^a = \sum_\nu {N_{\lambda\nu}}^\mu {n_{\nu 1}}^1 \ . \tag{3.2}$$

In particular the identity matrix $n_\rho = I$ associated with the trivial representation has norm 1. To the Gram matrix of scalar products one may then apply the Schmidt orthogonalisation method to determine a basis of vectors, $\widetilde{n}_\lambda$, which are obtained from the original n_λ by a triangular integer valued matrix, starting from $\tilde{n}_\rho = n_\rho = I$, and which are mutually orthogonal. In the most favourable cases, such as those presently discussed, all these orthogonal basis vectors have a norm equal to 1. This means that $\sum_a \left({\widetilde{n}_{\lambda 1}}{}^a \right)^2 = 1$, hence there exists a unique vertex a_λ contributing to this matrix element: ${\widetilde{n}_{\lambda 1}}{}^a = \delta_{a\, a_\lambda}$. (If the squared norms of some $\widetilde{n}$ are equal to 2 or 3, one has rather a sum of two or three Kronecker deltas, while squared norms larger or equal to 4 present new options that have to be examined in turn: either some entries ${\widetilde{n}_{\lambda 1}}{}^a$ are larger than 1, or there are more vertices contributing to them, see [5] for examples.) The above triangular system induces an order $\prec$ between the labels of the basis and one may invert it to determine the original n_λ as a sum of such contributions

$${n_{\lambda 1}}^a = \delta_{a a_\lambda} + \sum_{\mu \prec \lambda} b_{\lambda\mu} \delta_{a a_\mu}, \qquad b_{\lambda\mu} = {n_{\lambda 1}}^{a_\mu} \in \mathbb{Z}_{\geq 0} \ . \tag{3.3}$$

The set of vertices $\mathcal{V}$ is made of the a_λ, where λ labels the basis $\widetilde{n}_\lambda$.

3. In the last step one reconstructs the whole NIM-rep from the knowledge of the entries ${n_{\lambda 1}}^a$. This is done recursively. Assume first for simplicity that all $\widetilde{n}_\lambda$ in step 2 have norm 1. Take for example any fundamental representation called here $\bar\Lambda$ generically, suppose all entries ${n_{\bar\Lambda a_\mu}}^a$ have been determined for $\mu \prec \lambda$ and evaluate in two different ways $(n_\lambda n_{\bar\Lambda})_1{}^a = {n_{\bar\Lambda a_\lambda}}^a + \sum_{\mu\prec\lambda} b_{\lambda\mu} {n_{\bar\Lambda a_\mu}}^a = \sum_\nu {N_{\lambda\bar\Lambda}}^\nu {n_{\nu\,1}}^a$, a relation which determines ${n_{\bar\Lambda a_\lambda}}^a$. If some $\widetilde{n}_\lambda$ has a norm > 1 and ${n_{\lambda 1}}^a = \sum_\alpha \delta_{a a_{\lambda_\alpha}}$, this leads to dichotomic choices, as we have to split the sum $\sum_\alpha n_{\bar\Lambda a_{\lambda_\alpha}}$ into individual matrices, but at any rate, this is a finite problem.

This procedure has been applied successfully to the problem at hand. In step 1, one finds again a dependence on the parity of N and/or of h. While for odd $N = 2l+1$ and h odd, ${n_{\lambda\,1}}^1 = 1$ for all λ in the alcove, for h even, ${n_{\lambda\,1}}^1 \neq 0$ and in fact $= 1$ iff $\lambda = (\lambda_1, \cdots, \lambda_{2l})$ has all its Dynkin labels λ_i odd; for even $N = 2l$, the same happens iff $\lambda = (1, \lambda_2, 1, \lambda_4, \cdots, \lambda_{2l-2}, 1)$, i.e. when λ has Dynkin indices of odd rank equal to 1. In step 2, one may find a basis of orthonormal vectors, i.e. all norms are 1. Finding this basis may be laborious, and some external information, like the one coming from the embedding picture discussed in the previous section, is helpful but not necessary. Last step 3 then proceeds in a straightforward way, and the whole algorithm has been implemented in Mathematica.

The algorithm amounts effectively to solving recursively the two sets of equations (1.5), (1.9). To make contact with [7] note that the graph can be alternatively described identifying the vertices with the matrices $X_a := \sum_i {n_{\lambda 1}}^a \, n_\lambda$, which satisfy an equation analogous to (1.9), $n_\lambda\, X_a = \sum_b {n_{\lambda a}}^b \; X_b$, with non-trivial X_1. Giving a solution for the coefficients ${n_{\lambda 1}}^1$ determines X_1 or the algebra A of [7], and the graph is recovered starting from X_1 and using recursively as above the system (1.5).

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References

[1] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, Phys. Lett. **B 444** (1998) 163-166, hep-th/9809097; Nucl. Phys. **B 579** [FS] (2000) 707-773, hep-th/9908036.

[2] P. Di Francesco and J.-B. Zuber, Nucl. Phys. **B 338** [FS] (1990) 602-646.

[3] A. Ocneanu, *The classification of subgroups of quantum $SU(N)$*, Lectures at Bariloche Summer School, Argentina, Jan 2000, to appear in *AMS Contemporary Mathematics*, R. Coquereaux, A. Garcia and R. Trinchero, eds.

[4] T. Quella, *Branching rules of semi-simple Lie algebras using affine extensions*, `math-ph/0111020`; A. Alekseev, S. Fredenhagen, T. Quella and V. Schomerus, in preparation.

[5] F. Xu, Comm. Math. Phys. **192** (1998) 349-403.

[6] J. Böckenhauer and D.E. Evans, Comm. Math. Phys. **200** (1999) 57-103, `hep-th/9805023`; Comm. Math. Phys. **205** (1999) 183-228,`hep-th/9812110`;
J. Böckenhauer, D. E. Evans and Y. Kawahigashi, Comm. Math. Phys. **210** (2000) 733-784, `math.OA/9907149`.

[7] A.N. Kirillov and V. Ostrik, *On q-analog of McKay correspondence and ADE classification of $\widehat{sl}_2$ conformal field theories*, `math-ph/0101219`;
V. Ostrik, *Module categories, weak Hopf algebras and modular invariants* , `math.QA/0111139`.

[8] J. Fuchs, I. Runkel and C. Schweigert, *Conformal boundary conditions and 3D topological field theory* , `hep-th/0110158`.